Plane wave propagation

Time-harmonic fields

$$\nabla \cdot \tilde{E} = \tilde{\rho}_v / \epsilon$$
$$\nabla \times \tilde{E} = -j\omega\mu \tilde{H}$$
$$\nabla \cdot \tilde{H} = 0$$
$$\nabla \times \tilde{H} = \tilde{J} + j\omega\epsilon \tilde{E}$$

Complex permittivity

$$\nabla \times \tilde{H} = \tilde{J} + j\omega\epsilon \tilde{E} = (\sigma + j\omega\epsilon)\tilde{E} = j\omega\left(\epsilon - j\frac{\sigma}{\omega}\right)\tilde{E}$$

$$\epsilon_c = \epsilon - j\frac{\sigma}{\omega} = \epsilon' - j\epsilon''$$

$$\nabla \times \tilde{H} = j\omega\epsilon_c \tilde{E}$$

$$\nabla \cdot \nabla \times \tilde{H} = 0 = \nabla \cdot (j\omega\epsilon_c \tilde{E}) \text{, so } \nabla \cdot \tilde{E} = 0 \rightarrow \tilde{\rho}_v = 0$$

now we have:

$$\nabla \cdot \tilde{E} = 0$$
$$\nabla \times \tilde{E} = -j\omega\mu \tilde{H}$$
$$\nabla \cdot \tilde{H} = 0$$
$$\nabla \times \tilde{H} = j\omega\epsilon_c \tilde{E}$$

$$\nabla \times (\nabla \times \tilde{E}) = -j\omega\mu(\nabla \times \tilde{H}) \quad (\text{take curl of } \nabla \times \tilde{E} = -j\omega\mu\tilde{H})$$

$$\nabla \times \tilde{H} = j\omega\epsilon\tilde{E}$$

$$\nabla \times (\nabla \times \tilde{E}) = \omega^2\mu\epsilon_c\tilde{E} = \nabla(\nabla \cdot \tilde{E}) - \nabla^2\tilde{E}$$

vector identity

since $\nabla \cdot \tilde{E} = 0$,

$$\boxed{\nabla^2\tilde{E} + \omega^2\mu\epsilon_c\tilde{E} = 0} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

homogeneous wave equation for E

$\gamma^2 = -\omega^2\mu\epsilon_c$ $\qquad$ γ = propagation constant

$$\underline{\nabla^2\tilde{E} - \gamma^2\tilde{E} = 0}$$

similarly, $\underline{\nabla^2\tilde{H} - \gamma^2\tilde{H} = 0}$

- Plane wave propagation in lossless media

nonconducting $\rightarrow$ $\sigma = 0$, $\epsilon_c = \epsilon$

$$\gamma^2 = -\omega^2\mu\epsilon$$

wavenumber $k = \omega\sqrt{\mu\epsilon}$

$$\gamma^2 = -k^2$$

$$\nabla^2\tilde{E} + k^2\tilde{E} = 0$$

Uniform plane waves

$$\tilde{E} = \hat{x}\tilde{E}_x + \hat{y}\tilde{E}_y + \hat{z}\tilde{E}_z \qquad \nabla^2\tilde{E} + k^2\tilde{E} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E}_x = 0$$, similar for $\tilde{E}_y$, $\tilde{E}_z$

Uniform plane waves : E and H are uniform across infinite plane

assume uniform in xy plane

$$\frac{\partial \tilde{E}_x}{\partial x} = \frac{\partial \tilde{E}_y}{\partial y} = 0 \longrightarrow \frac{\partial^2 \tilde{E}_x}{\partial z^2} + k^2 \tilde{E}_x = 0$$, similar for E_y, H_x, H_y

$$\tilde{E}_z = \tilde{H}_z = 0 \quad \text{from} \quad \underbrace{\frac{\partial \tilde{H}_y}{\partial x}}_{=0} - \underbrace{\frac{\partial \tilde{H}_x}{\partial y}}_{=0} = j\omega\epsilon \tilde{E}_z$$

- No E or H along direction of propagation : TEM wave

$$\tilde{E}_x(z) = \tilde{E}_x^+(z) + \tilde{E}_x^-(z) = E_{x0}^+ e^{-jkz} + E_{x0}^- e^{jkz}$$

just like in transmission lines

Assume one component $\tilde{E}(z) = \hat{x}\tilde{E}_x^+(z) = \hat{x}\tilde{E}_{x0}^+ e^{-jkz}$

find H

$$\nabla \times \tilde{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu\left(\hat{x}\tilde{H}_x + \hat{y}\tilde{H}_y + \hat{z}\tilde{H}_z\right)$$

$$\tilde{H}_x = 0$$

$$\tilde{H}_y = \frac{1}{-j\omega\mu}\frac{\partial E_x^+(z)}{\partial z}$$

$$H_z = \frac{1}{-j\omega\mu}\frac{\partial E_x^+(z)}{\partial z} = 0$$

$$\tilde{H}_y(z) = \frac{k}{\omega\mu}\tilde{E}_{x0}^+ e^{-jkz} = \tilde{H}_{y0}^+ e^{-jkz}$$

$$\frac{\tilde{E}_{x0}^+}{\tilde{H}_{y0}^+} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$ intrinsic impedance

$= 377\,\Omega$ for free space

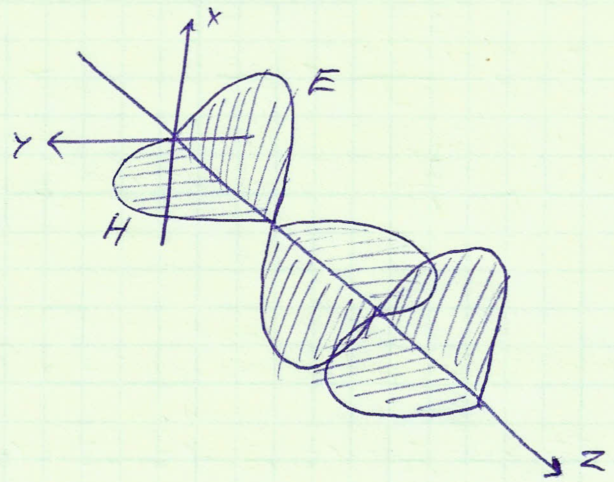
More on plane wave propagation

$$u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} = 3\times10^8 \text{ m/s for free space}$$

$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f}$$

Note that E and H are <u>in phase</u> with eachother

they reach maximum and minimum at the same time, location



General relation between E and H

$$\tilde{H} = \frac{1}{\eta}\hat{k}\times\tilde{E}$$

$$\tilde{E} = -\eta\hat{k}\times\tilde{H}$$

Also $\tilde{E}\times\tilde{H}$ gives direction of $\hat{k}$